DISCRETE MATHEMATICS 10 (1974) 9–13. © North-Holland Publishing Company

AN UPPER BOUND FOR THE DIAMETER OF A POLYTOPE *

David BARNETTE

Department of Mathematics, University of California, Davis, Calif. 95616, USA

Received 5 March 1974

Abstract. The distance between two vertices of a polytope is the minimum number of edges in a path joining them. The diameter of a polytope is the greatest distance between two vertices of the polytope. We show that if P is a d-dimensional polytope with n facets, then the diameter of P is at most $\frac{1}{3}2^{d-3}(n-d+\frac{5}{2})$.

1. Introduction

If we have two vertices of a convex polytope, the number of edges in a shortest path joining them is called the *distance* between the vertices. The maximum distance between pairs of vertices is called the *diameter* of the polytope.

Because of applications in linear programming, it is of interest to find an upper bound for the diameter of a d-dimensional polytope in terms of d and n, the number of facets of P. This has been a difficult problem and the results so far are far from the conjectured upper bounds (see [1–3]). The best general result which applies to all dimensions is that the diameter is at most $2^{d-3}n$ (see [3]). For certain small dimensions and for polytopes with few vertices, better results are known [1–3].

In this paper we improve on the bound $2^{d-3}n$ by showing that the diameter is less than or equal to $\frac{1}{3}2^{d-3}(n-d+\frac{5}{2})$.

2. Definitions

A *d-polytope* is a d-dimensional set that is the convex hull of a finite

* Research supported by a Sloan Foundation grant.

set of points. A *face* of a d-polytope P is the intersection of P with a supporting hyperplane. Faces of dimension $d-1$ are called *facets* and faces of dimension $d-2$ are called *subfacets*. A d-polytope is *simple* provided each vertex meets exactly d edges.

The ***graph*** of a polytope is the graph consisting of its vertices and edges. A graph is *d-polyhedral* provided it is isomorphic to the graph of some d-polytope. A characterization of 3-polyhedral graphs is given by Steinitz [4]:

A graph (without loops or multiple edges) is 3-polyhedral if and only if it is planar and 3-connected.

If G is a graph in the plane Π, then the connected components of $\Pi \sim G$ are called the *faces* of G. If G is 3-polyhedral, then its faces correspond in a natural way to the facets of the 3-polytope. It follows from Steinitz' theorem that if G is a graph in the plane, with more than three vertices, and without multiple edges, such that each face is bounded by a triangle, then the graph and its dual are 3-polyhedral.

By a ***path*** in a d-polytope P we mean a simple arc in P consisting of edges of P. The ***length*** of the path is the number of edges in it. If x and y are vertices of P, then $\delta(x,y)$ is defined to be the minimum length of paths from x to y and is called the *distance* from x to y. If A is a face of P, then $\delta(x,A)$ is defined to be min $\delta(x,y)$ taken over all vertices y of A, and is called the *distance* from x to A. If F is a face of P containing x and y, then $\delta_F(x,y)$ is defined to be the length of a shortest path in F joining x and y. The notation $\delta_F(x,A)$ is similarly defined when A is a face of F. The ***diameter*** of a polytope P is max $\delta(x,y)$ taken over all pairs of vertices (x,y) in P. If x and y are vertices of a d-polytope P, then a ***chain*** joining x and y is a sequence of facets $F_1, \ldots, F_k$ such that $F_i \cap F_{i+1}$ is a facet of both F_i and F_{i+1}, $x \in F_1$ and $y \in F_k$. The ***length*** of a chain is the number of facets in it. A chain of length k is called a ***k-chain***.

There is a process for constructing smaller 3-polyhedral graphs from a given 3-polyhedral graph, called removing edges, which will be useful. Let e be an edge of the graph G of a simple 3-polytope. Let v_1 and v_2 be the vertices of e, let v_3 and v_4 be the other vertices joined to v_1 and let v_5 and v_6 be the vertices joined to v_2 (these may not all be distinct). Let G' be the graph consisting of all vertices and edges of G that miss e, together with edges joining v_3 to v_4 and v_5 to v_6. We say that G' is obtained from G by ***removing e***. If G' is also the graph of a 3-polytope, we say that e is *removable*.

3. The main result

Lemma 1. *If G is the graph of a simple 3-polytope P, and if P is not a tetrahedron, then each face of G has a removable edge.*

Proof. Let G^* be the dual graph of G. Removing an edge of G corresponds to shrinking an edge of G^* to a vertex. In G^* every face is bounded by a triangle. When we shrink an edge e of G^*, every face in the new graph will be bounded by a triangle and the new graph will be without multiple edges unless P is a tetrahedron or e belongs to a triangle in G^* that is not a face of G^*. Such triangles will be called *non-facial* triangles. Let F be a face of G and let v be the corresponding vertex in G^*. Let the edges meeting v be $e_1, ..., e_n$. If no e_i belongs to a non-facial triangle, we are done. Otherwise, we choose a non-facial triangle T containing v enclosing a minimum number of vertices joined to v. Let v_1 be any vertex joined to v that is enclosed by T. The edge e_i joining v and v_1 cannot belong to a non-facial triangle because such a triangle cannot pass outside of T, and T is minimal. The edge of G corresponding to e_i will be the removable edge of F.

Lemma 2. *If x and y are vertices of a simple 3-polytope P with $\delta(x,y) = k$, and if there are exactly n facets F of P such that $\delta(x, F) < k$, then $k \leqslant \frac{2}{3}n - 1$.*

Proof. First, suppose that all facets F of P are such that $\delta(x, F) < k$. In this case, a theorem of Klee [11] which states that a 3-polytope with n facets has diameter at most $\frac{2}{3}n - 1$, applies. Suppose, now, that some facet F is such that $\delta(x, F) \geqslant k$. By Lemma 1, there is a removable edge e of F. The edge e cannot belong to any path of length $\leqslant k$ containing x, thus when we remove e to produce the graph of a 3-polytope P', no facet of P of distance $<k$ from x becomes a facet of distance $\geqslant k$ in P'. Furthermore, $\delta_{P'}(x, y) = \delta_P(x, y)$.

We may continue removing edges until there are no more facets of distance $\geqslant k$, and now the first part of our argument applies.

Theorem 1. *If x_0 and y are vertices of a d-polytope P, $d \geqslant 3$, with $\delta(x_0, y) = \delta$, and if there are no more than n facets of P of distance at most $\delta - 1$ from x_0, then $\delta \leqslant \frac{1}{3}(2^{d-2})(n-d+\frac{5}{2})$.*

Proof. Our proof is by induction on d. Lemma 2 begins our induction at $d = 3$. Suppose, now, that P is a d-polytope, $d > 3$. Let k be the length of a shortest chain from x_0 to y and let F_1 be the first facet of such a chain. Among all k-chains from x_0 to y beginning with F_1, choose one with a second facet F_2 such that $\delta_{F_1}(x_0, F_1 \cap F_2)$ is minimal. Let x_1 be a vertex of $F_1 \cap F_2$ such that $\delta_{F_1}(x_0, x_1)$ is minimal. Among all k-chains joining x_0 and y beginning with F_1 and F_2, choose one with third facet F_3 such that $\delta_{F_2}(x_1, F_2 \cap F_3)$ is minimal. Choose a vertex x_2 of $F_2 \cap F_3$ such that $\delta_{F_2}(x_1, x_2)$ is minimal, and so on. Continuing in this way, we construct a chain $F_1, F_2, ..., F_k$ from x_0 to y. We define a subfacet S of P on a facet F_i to be a *close* subfacet provided $\delta_{F_i}(x_{i-1}, S) < \delta_{F_i}(x_{i-1}, x_i)$. Let Γ_i be a shortest path from x_{i-1} to x_i in F_i. The union of the Γ_i's is a path Γ in P.

We now examine facets F of P that are not in $F_1, ..., F_k$, and such that F intersects some F_i on a close subfacet. If such a facet met $F_1, ..., F_k$ on two F_i's that are separated by more than one facet in the chain, then k would not be minimal. If F meets F_i and F_{i+2}, for some i, such that $F_i \cap F$ is a close facet, then we can replace F_{i+1} by F and contradict the way that F_{i+1} was chosen, because $\delta_{F_i}(x_{i-1}, F_i \cap F_{i+1}) > \delta_{F_i}(x_{i-1}, F_i \cap F)$. Thus each facet that meets the chain and is not in it, intersects the chain on at most two close subfacets.

On the other hand, if $\delta(F, x_0) \geqslant \delta$, then F will not intersect the chain on any close subfacet.

Let S_i be the number of close subfacets on F_i. The above argument shows that

$$\sum_{i=1}^{k} S_i \leqslant 2(n-k) + 2(k-1) = 2n-2.$$

That is, the total number of close subfacets is at most two times the number of facets that intersect the chain on close subfacets, plus the number of close subfacets which are the intersection of two consecutive facets in the chain.

By induction, the length of each Γ_i is at most $\frac{1}{3}(2^{d-3})(S_i - d + 1 + \frac{5}{2})$. Thus the length of Γ is at most

$$\sum_{i=1}^{k} \tfrac{1}{3}(2^{d-3})(S_i - d + 1 + \tfrac{5}{2}) = \tfrac{1}{3}(2^{d-3})\left(\left(\sum_{i=1}^{k} S_i\right) - kd + k + \tfrac{5}{2}k\right)$$

$$\leqslant (\tfrac{1}{3}(2^{d-3}))((2n-2) - 2d + 2 + 5)$$

$$= \tfrac{1}{3}2^{d-2}(n - d + \tfrac{5}{2}) \qquad \text{when } k \geqslant 2\,.$$

If $k = 1$, then the length of Γ is at most

$$\tfrac{1}{3}(2^{d-3})(n-d+1+\tfrac{5}{2}) = \tfrac{1}{3}(2^{d-3})(n-d+\tfrac{5}{2}) \leqslant \tfrac{1}{3}2^{d-2}(n-d+\tfrac{5}{2}).$$

This follows because x_0 and y lie on a common facet F which is a $(d-1)$-polytope and any close subfacet on F is the intersection of F with a facet of distance less than $\delta(x_0, y)$ from x_0. Since $\delta(x_0, F) = 0$, there can be at most $n-1$ close subfacets on F, and induction gives the above inequalities.

Corollary. *The diameter of a d-polytope, $d \geqslant 3$, with n facets is at most* $\tfrac{1}{3}2^{d-3}(n-d+\tfrac{5}{2})$.

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